DIRAC EQUATION

in 2-dimensional spacetime

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Introduction. Expository special relativity is, for many purposes, well served by the pretense that we inhabit a spacetime of $1+1$ dimensions, though in such a world proper rotation is impossible, boosts are necessarily colinear, and certain kinematic phenomena—Thomas precession, most notably—remain hidden from view.

The Schrödinger equation came into the world wearing 3-dimensional dress, intent upon serious physical business (physics of the hydrogen atom); the "quantum theory of 1-dimensional systems" was a pedagogical afterthought. Similarly serious were Dirac's motivating objectives (theory of the electron, relativistically corrected theory of hydrogen, clarification of spin concept), but in the latter instance the 3-dimensionality of space has seemed so central to the architecture of the theory that no tradition of "pedagogical pull-back to lower dimension" has come into being. I am motivated to inquire into the question of whether or not such a "toy Dirac theory" has things to teach us. We will find that it has, in fact, *many* valuable lessons to impart, and that it speaks of deep things with engaging simplicity.

Suppose, therefore, that we are relativistically informed one-dimensional physicists who, in reference to our inertial frame (and in the absence of gravity), write

$$\boldsymbol{g} \equiv \|g_{\mu\nu}\| \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

to describe the Lorentzian metric structure of spacetime. It has been our habit to erect our theory of the wave equation $\Box\psi = 0$ on a "factorization trick"

$$\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu = \partial_0^2 - \partial_1^2 = (\partial_0 - \partial_1)(\partial_0 + \partial_1)$$

and we have noticed, by the way, that the trick fails when we attempt to apply it to the Klein-Gordon equation $(\Box + \varkappa^2)\psi = 0$.

Proceeding in imitation of Dirac, we observe that the wave operator can be written as a square

$$\Box = (\boldsymbol{\gamma}^\mu\partial_\mu)(\boldsymbol{\gamma}^\nu\partial_\nu) \tag{2}$$

provided $\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu = 2g^{\mu\nu}\mathbf{I}$, of which

$$\boldsymbol{\gamma}^0\boldsymbol{\gamma}^0 = \mathbf{I}, \quad \boldsymbol{\gamma}^0\boldsymbol{\gamma}^1 + \boldsymbol{\gamma}^1\boldsymbol{\gamma}^0 = \mathbf{0}, \quad \boldsymbol{\gamma}^1\boldsymbol{\gamma}^1 = -\mathbf{I} \tag{3}$$

supply a more explicit account. The Klein-Gordon operator can then be rendered

$$\Box + \varkappa^2 = (\boldsymbol{\gamma}^\mu \partial_\mu - i\varkappa)(\boldsymbol{\gamma}^\mu \partial_\mu + i\varkappa) \tag{4}$$

and we are led to the Dirac equation

$$(\boldsymbol{\gamma}^\mu \partial_\mu + i\varkappa)\psi = 0 \tag{5}$$

The objects $\boldsymbol{\gamma}^\mu$ cannot, by (3), be real/complex numbers. A pair of 2×2 *matrices* that do the trick are

$$\boldsymbol{\gamma}^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}^1 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6}$$

Notice that while $\boldsymbol{\gamma}^0$ is hermitian, $\boldsymbol{\gamma}^1$ is antihermitian. The hermitian matrix

$$\boldsymbol{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ which, as it happens, is just } \boldsymbol{\gamma}^0 \tag{7}$$

is, however, available as a "hermitianizer" in the sense[1] that it renders $\boldsymbol{G}\boldsymbol{\gamma}^0$ and $\boldsymbol{G}\boldsymbol{\gamma}^1$ simultaneously hermitian.

Looking back now to (5) we see that ψ has become a 2-component wave function, and that were we wrote $i\varkappa$ we might more properly have written $i\varkappa\,\mathbf{I}$. The "toy Dirac equation" (5) is a coupled pair of equations, which can be spelled out

$$\begin{pmatrix} i\varkappa & \partial_0 - \partial_1 \\ \partial_0 + \partial_1 & i\varkappa \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8}$$

Multiplication by the conjugated operator yields two copies of the Klein-Gordon equation:

$$\begin{pmatrix} -i\varkappa & \partial_0 - \partial_1 \\ \partial_0 + \partial_1 & -i\varkappa \end{pmatrix} \begin{pmatrix} i\varkappa & \partial_0 - \partial_1 \\ \partial_0 + \partial_1 & i\varkappa \end{pmatrix} = \begin{pmatrix} \Box + \varkappa^2 & 0 \\ 0 & \Box + \varkappa^2 \end{pmatrix}$$

Lagrangian formulation. It is (see CFT, Chapter 2, p. 24) the existence of a hermitianizer that provides access to the methods of Lagrangian field theory. We introduce

$$\begin{aligned} \mathcal{L} &= -\hbar c\Big[i\tfrac{1}{2}\big\{\tilde\psi_\mu \boldsymbol{\gamma}^\mu \psi - \tilde\psi\,\boldsymbol{\gamma}^\mu \psi_\mu\big\} + \varkappa\,\tilde\psi\,\psi\Big] \\ &= \hbar c\left[\frac{\tilde\psi_\mu \boldsymbol{\gamma}^\mu \psi - \tilde\psi\,\boldsymbol{\gamma}^\mu \psi_\mu}{2i} - \varkappa\,\tilde\psi\,\psi\right] \end{aligned} \tag{9.0}$$

with $\tilde\psi \equiv \psi^\dagger \boldsymbol{G}$ and $\psi_\mu \equiv \partial_\mu \psi$; the $\hbar c$-factor has been introduced in order to ensure that $[\mathcal{L}] = \text{energy/length}$, and in the presumption that $[\tilde\psi\,\psi] = 1/\text{length}$,

[1] Compare (2–55) in CLASSICAL FIELD THEORY (1999). I will henceforth write CFT when referring to that source.

while the minus sign is physically inconsequential/cosmetic. Gauge-equivalent to $\mathcal{L}$ are

$$\mathcal{L}_1 \equiv \mathcal{L} + \tfrac{1}{2}i\hbar c\partial_\mu(\tilde{\psi}\boldsymbol{\gamma}^\mu\psi) = +\hbar c\big[\tilde{\psi}(i\boldsymbol{\gamma}^\mu\psi_\mu - \varkappa\psi)\big] \tag{9.1}$$

$$\mathcal{L}_2 \equiv \mathcal{L} - \tfrac{1}{2}i\hbar c\partial_\mu(\tilde{\psi}\boldsymbol{\gamma}^\mu\psi) = -\hbar c\big[(\tilde{\psi}_\mu\boldsymbol{\gamma}^\mu i + \tilde{\psi}\varkappa)\psi\big] \tag{9.2}$$

which permit one to write

$$\mathcal{L} = \tfrac{1}{2}(\mathcal{L}_1 + \mathcal{L}_2)$$

The $\tilde{\psi}^{\text{th}}$ Lagrange derivative operator looks most simply to (9.1), and yields the Dirac equation (5) as a field equation. The ψ^{th} Lagrange derivative of (9.2) yields as the other field equation $\tilde{\psi}_\mu\boldsymbol{\gamma}^\mu i + \tilde{\psi}\varkappa = 0$, which is seen by the following little argument

$$\begin{aligned}(\tilde{\psi}_\mu\boldsymbol{\gamma}^\mu i + \tilde{\psi}\varkappa) &= i(\tilde{\psi}_\mu\boldsymbol{\gamma}^\mu - i\tilde{\psi}\varkappa)\\ &= i(\boldsymbol{\gamma}^\mu\psi_\mu + i\varkappa\psi)^\dagger\boldsymbol{G}\end{aligned}$$

to be in effect the conjugate transpose of (5).

From the simple design of (9.1) it follows that if ψ is a *solution* of the Dirac equation (5) then $\mathcal{L}_1 = 0$, and by quick extension of that argument we learn that

$$\mathcal{L},\ \mathcal{L}_1 \text{ and } \mathcal{L}_2 \text{ vanish numerically if } \psi \text{ satisfies the Dirac equation} \tag{10}$$

From the manifest reality of $\mathcal{L}$ it follows by Noether's theorem[2] that

$$\partial_\mu Q^\mu = 0 \quad\text{where}\quad Q^\mu \equiv c\tilde{\psi}\boldsymbol{\gamma}^\mu\psi \tag{11}$$

where the factor c has been introduced so as to achieve $[Q^\mu] = 1/\text{time}$ (which in 2-dimensional spacetime is the dimension of "number flux"). Direct verification of (11) is easily accomplished:

$$\begin{aligned}\partial_\mu Q^\mu &= c\tilde{\psi}_\mu\boldsymbol{\gamma}^\mu\psi + c\tilde{\psi}\boldsymbol{\gamma}^\mu\psi_\mu\\ &= c(\tilde{\psi}i\varkappa)\psi + c\tilde{\psi}(-i\varkappa\psi) \quad \text{by the field equations}\\ &= 0\end{aligned}$$

Recalling the definitions (6) and (7), we have these explicit formulae:

$$\left.\begin{aligned}Q^0 &= \psi^{*0}\psi^0 + \psi^{*1}\psi^1\\ Q^1 &= \psi^{*0}\psi^0 - \psi^{*1}\psi^1\end{aligned}\right\} \tag{12}$$

Noether has supplied this generic description of the stress-energy tensor:[3]

$$S^\mu{}_\nu = \tilde{\psi}_\nu\frac{\partial\mathcal{L}}{\partial\tilde{\psi}_\mu} + \frac{\partial\mathcal{L}}{\partial\psi_\mu}\,\psi_\nu - \underbrace{\mathcal{L}}_{\text{drop, in consequence of (10)}}\delta^\mu{}_\nu$$

[2] See CFT (3–69).
[3] See CFT (1–34).

Working from (9.0), we obtain

$$S^\mu{}_\nu = \hbar c\left[\frac{\tilde\psi_\nu\boldsymbol{\gamma}^\mu\psi - \tilde\psi\,\boldsymbol{\gamma}^\mu\psi_\nu}{2i}\right] \tag{13}$$

in connection with which we observe that

$$\begin{aligned}\partial_\mu S^\mu{}_\nu &\sim \tilde\psi_{\mu\nu}\boldsymbol{\gamma}^\mu\psi + \tilde\psi_\nu\boldsymbol{\gamma}^\mu\psi_\mu - \tilde\psi_\mu\boldsymbol{\gamma}^\mu\psi_\nu - \tilde\psi\,\boldsymbol{\gamma}^\mu\psi_{\mu\nu}\\ &= (\tilde\psi i\varkappa)_\nu\psi + \tilde\psi_\nu(-i\varkappa\,\psi) - (\tilde\psi i\varkappa)\psi_\nu - \tilde\psi(-i\varkappa\,\psi)_\nu \text{ by field equations}\\ &= 0\end{aligned}$$

Notice that $S_{\mu\nu}$ is *not symmetric*: in this respect also toy Dirac theory is found to mimic precisely its 4-dimensional prototype.[4] Had we worked from (9.1) or (9.2) we would have been led to distinct but similar results.

Lorentz covariance. In 2-dimensional spacetime we write

$$\Lambda: \quad x \longrightarrow X = \boldsymbol{\Lambda}x \quad \text{with} \quad \boldsymbol{\Lambda}^{\mathsf T}\boldsymbol{g}\,\boldsymbol{\Lambda} = \boldsymbol{g} \tag{14}$$

to describe a Lorentz transformation. Necessarily $\det\boldsymbol{\Lambda} = \pm 1$. Infinitesimal Lorentz transformations are necessarily proper, and can be described

$$\boldsymbol{\Lambda} = \mathbf{I} + \boldsymbol{\alpha} + \cdots \quad \text{with} \quad \boldsymbol{\alpha}^{\mathsf T}\boldsymbol{g} + \boldsymbol{g}\,\boldsymbol{\alpha} = \mathbf{0} \;\Rightarrow\; \boldsymbol{\alpha} = \delta\omega\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{15}$$

By iteration we obtain

$$\boldsymbol{\Lambda} = \exp\left\{\omega\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\} = \begin{pmatrix}\cosh\omega & \sinh\omega\\ \sinh\omega & \cosh\omega\end{pmatrix} \tag{16}$$

where $\beta = \tanh\omega$ serves (via $\cosh\omega = 1/\sqrt{1-\beta^2} \equiv \gamma$ and $\sinh\omega = \beta/\gamma$) to establish explicit contact with kinematic aspects of the theory of Lorentz transformations. We observe that in 2-dimensional spacetime all (proper) Lorentz transformations have the irrotational character of "boosts."

The components of multicomponent fields are assumed to fold among themselves in linear representation of the Lorentz group $O(1,1)$:[5]

$$\Lambda: \quad \psi \longrightarrow \Psi = \boldsymbol{U}(\Lambda)\,\psi \tag{17.1}$$

First partials of such a field therefore transform

$$\Lambda: \quad \psi_\mu \longrightarrow \Psi_\mu = \boldsymbol{U}(\Lambda)\,\Lambda^\mu{}_\nu\psi_\mu \tag{17.2}$$

[4] Compare CFT (2–99).
[5] For omitted details see CFT, Chapter 2, pp. 32 *et seq*.

The Lorentz invariance of the Dirac Lagrangian (9.0) can be shown to entail (compare CFT (2–70))

$$\boldsymbol{U}^{-1} = \boldsymbol{G}^{-1}\boldsymbol{U}^{\dagger}\boldsymbol{G} \tag{18.1}$$

$$\boldsymbol{U}^{-1}\boldsymbol{\gamma}^{\mu}\boldsymbol{U} = \Lambda^{\mu}{}_{\nu}\boldsymbol{\gamma}^{\nu} \tag{18.2}$$

The first of those conditions asserts the "$\boldsymbol{G}$-unitarity" of the 2×2 matrix $\boldsymbol{U}$, and would reduce to ordinary unitarity if it were the case that $\boldsymbol{G} = \mathbf{I}$. Write

$$\boldsymbol{U} = \mathbf{I} + \boldsymbol{\beta} + \cdots$$

Then (18.1) entails the "$\boldsymbol{G}$-antihermiticity" of $\boldsymbol{\beta}$ (i.e., that $\boldsymbol{G\beta} + (\boldsymbol{G\beta})^{\dagger} = \mathbf{0}$) while (18.2) gives

$$\boldsymbol{\gamma}^{\mu}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\gamma}^{\mu} = \alpha^{\mu\nu}\boldsymbol{\gamma}_{\nu} \tag{19}$$

Stealing now from I know not whom, I claim that $\boldsymbol{\beta}$ can therefore be described

$$\boldsymbol{\beta} = \tfrac{1}{8}\alpha^{\rho\sigma}(\boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}_{\sigma} - \boldsymbol{\gamma}_{\sigma}\boldsymbol{\gamma}_{\rho}) \tag{20}$$

and do so on the basis of the following argument: write $\boldsymbol{\gamma}^{\mu}(\boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}_{\sigma} - \boldsymbol{\gamma}_{\sigma}\boldsymbol{\gamma}_{\rho})$ and use $\boldsymbol{\gamma}^{\mu}\boldsymbol{\gamma}^{\nu} + \boldsymbol{\gamma}^{\nu}\boldsymbol{\gamma}^{\mu} = 2g^{\mu\nu}\mathbf{I}$, in the form $\boldsymbol{\gamma}^{\mu}\boldsymbol{\gamma}_{\rho} = 2\delta^{\mu}{}_{\rho}\mathbf{I} - \boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}^{\mu}$, to pull the $\boldsymbol{\gamma}^{\mu}$ factors through to the right; one obtains

$$\begin{aligned}
\boldsymbol{\gamma}^{\mu}(\boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}_{\sigma} - \boldsymbol{\gamma}_{\sigma}\boldsymbol{\gamma}_{\rho}) &= 2(\delta^{\mu}{}_{\rho}\boldsymbol{\gamma}_{\sigma} - \delta^{\mu}{}_{\sigma}\boldsymbol{\gamma}_{\rho}) - \boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}^{\mu}\boldsymbol{\gamma}_{\sigma} + \boldsymbol{\gamma}_{\sigma}\boldsymbol{\gamma}^{\mu}\boldsymbol{\gamma}_{\rho} \\
&= 2(\delta^{\mu}{}_{\rho}\boldsymbol{\gamma}_{\sigma} - \delta^{\mu}{}_{\sigma}\boldsymbol{\gamma}_{\rho}) - 2(\boldsymbol{\gamma}_{\rho}\delta^{\mu}{}_{\sigma} - \boldsymbol{\gamma}_{\sigma}\delta^{\mu}{}_{\rho}) \\
&\qquad + \boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}_{\sigma}\boldsymbol{\gamma}^{\mu} - \boldsymbol{\gamma}_{\sigma}\boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}^{\mu}
\end{aligned}$$

giving

$$\begin{aligned}
\alpha^{\rho\sigma}\big[\boldsymbol{\gamma}^{\mu}(\boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}_{\sigma} - \boldsymbol{\gamma}_{\sigma}\boldsymbol{\gamma}_{\rho}) - (\boldsymbol{\gamma}_{\rho}\boldsymbol{\gamma}_{\sigma} - \boldsymbol{\gamma}_{\sigma}\boldsymbol{\gamma}_{\rho})\boldsymbol{\gamma}^{\mu}\big] &= \alpha^{\rho\sigma}\big[4(\delta^{\mu}{}_{\rho}\boldsymbol{\gamma}_{\sigma} - \delta^{\mu}{}_{\sigma}\boldsymbol{\gamma}_{\rho})\big] \\
&= 4(\alpha^{\mu\nu} - \alpha^{\nu\mu})\,\boldsymbol{\gamma}_{\nu} \\
&= 8\alpha^{\mu\nu}\boldsymbol{\gamma}_{\nu}
\end{aligned}$$

which completes the demonstration. Looking back again to the definitions (1), (6) and (15) of $\boldsymbol{g}$, $\boldsymbol{\gamma}^{0}$, $\boldsymbol{\gamma}^{1}$ and $\|\alpha^{\mu}{}_{\nu}\|$ we obtain

$$\boldsymbol{\gamma}_{0} = g_{0\beta}\boldsymbol{\gamma}^{\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}_{1} = g_{1\beta}\boldsymbol{\gamma}^{\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\|\alpha^{\mu\nu}\| = \|\alpha^{\mu}{}_{\beta}\|\|g^{\beta\nu}\| = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

which when brought to (20) give

$$\boldsymbol{\beta} = \tfrac{1}{2}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \tag{21}$$

Calculation confirms that the $\boldsymbol{\beta}$ thus described does in fact satisfy (19). By trivial iteration we have

$$\boldsymbol{U} = \exp\left\{\tfrac{1}{2}\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix}\right\} = \begin{pmatrix}e^{+\frac{1}{2}\omega} & 0\\ 0 & e^{-\frac{1}{2}\omega}\end{pmatrix} \tag{22}$$

Calculation gives

$$\begin{aligned}\boldsymbol{U}^{-1}\boldsymbol{\gamma}^0\boldsymbol{U} &= \begin{pmatrix}0 & e^{-\omega}\\ e^{+\omega} & 0\end{pmatrix} = \cosh\omega\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \sinh\omega\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\ \boldsymbol{U}^{-1}\boldsymbol{\gamma}^1\boldsymbol{U} &= \begin{pmatrix}0 & -e^{-\omega}\\ e^{+\omega} & 0\end{pmatrix} = \sinh\omega\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \cosh\omega\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\end{aligned}$$

in exact agreement with (18.2).

Let us look now more closely to the implications of (18.1). Necessarily

$$\det\boldsymbol{U} = e^{i\phi} \tag{23.1}$$

so $\boldsymbol{G}$-unitary matrices can (in the 2-dimensional case) be displayed

$$\boldsymbol{U} = e^{\frac{1}{2}i\phi}\cdot\boldsymbol{S} \tag{23.2}$$

where $\boldsymbol{S}$ is a *unimodular* $\boldsymbol{G}$-unitary matrix:

$$\boldsymbol{S}^{-1} = \boldsymbol{G}^{-1}\boldsymbol{S}^\dagger\boldsymbol{G} \quad\text{with}\quad \det\boldsymbol{S} = 1 \tag{23.3}$$

The most general such 2×2 matrix is found to have the form

$$\boldsymbol{S} = \begin{pmatrix}a & ib\\ ic & d\end{pmatrix} \quad\text{with}\quad a, b, c, d \text{ real and } ad + bc = 1 \tag{23.4}$$

If we write $\boldsymbol{S} = e^{i\boldsymbol{H}}$ then $\boldsymbol{S}$ will be unimodular if and only if $\text{tr}\boldsymbol{H} = 0$, and $\boldsymbol{G}$-unitary if and only if $\boldsymbol{H}$ is $\boldsymbol{G}$-hermitian

$$\boldsymbol{H} = \boldsymbol{G}^{-1}\boldsymbol{H}^\dagger\boldsymbol{G} \tag{23.5}$$

which requires that $\boldsymbol{GH}$ be hermitian in the standard sense; i.e., that

$$\boldsymbol{H} = \boldsymbol{G}^{-1}\cdot\begin{pmatrix}p & r+is\\ r-is & q\end{pmatrix} = \begin{pmatrix}r-is & q\\ p & r+is\end{pmatrix} \tag{24.6}$$

which will be traceless if and only if $r = 0$. Looking in the light of these formal developments back to (22), we see that the $\boldsymbol{U}$ encountered there is unimodular—an instance of (23.4) with $b = c = 0$ and $d = a^{-1}$.

We have arrived at the association

$$\boldsymbol{\Lambda}(\omega) = \begin{pmatrix}\cosh\omega & \sinh\omega\\ \sinh\omega & \cosh\omega\end{pmatrix} \iff \boldsymbol{U}(\omega) = \begin{pmatrix}e^{+\frac{1}{2}\omega} & 0\\ 0 & e^{-\frac{1}{2}\omega}\end{pmatrix} \tag{25}$$

from which it is already clear that if Lorentz covariance of the toy Dirac theory is to be achieved then the 2-component wave function ψ cannot transform as a 2-vector, but must transform by the distinctive rule

$$\begin{aligned}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} \longrightarrow \begin{pmatrix}\Psi^1\\ \Psi^2\end{pmatrix} =&\boldsymbol{U}(\omega)\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}\\ \boldsymbol{U}(\omega) &= \begin{pmatrix}\cosh\frac{1}{2}\omega+\sinh\frac{1}{2}\omega & 0\\ 0 & \cosh\frac{1}{2}\omega-\sinh\frac{1}{2}\omega\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{\frac{1+\beta}{1-\beta}} & 0\\ 0 & \sqrt{\frac{1-\beta}{1+\beta}}\end{pmatrix}\end{aligned}$$

Notice that $\boldsymbol{U}$ has turned out to be unimodular, though we nowhere had reason to insist that it be so.

The intrusion of (what we might informally call) half-angles speaks to us of the familiar double-valuedness of the spinor representations of $O(3)$. . . and indeed: it follows transparently from (18) that if $\boldsymbol{U}$ satifies that pair of conditions, then so also does its negative. So in place of (25) we might more properly write

$$\boldsymbol{\Lambda}(\omega) \quad\Longleftrightarrow\quad \pm\boldsymbol{U}(\omega) \tag{26}$$

In the present context, however, the two branches of the *spinor representation of the Lorentz group* $O(1,1)$ are—uncharacteristically—disjoint. They can be connected, but at cost of a *complexification of the "rapidity" parameter*:

$$\boldsymbol{\Lambda}(\omega+2\pi i)=\boldsymbol{\Lambda}(\omega) \quad\text{but}\quad \boldsymbol{U}(\omega+2\pi i)=-\boldsymbol{U}(\omega) \tag{27}$$

We noticed that only a 1-parameter subgroup of the full 3-parameter group $SU(2;\boldsymbol{G})$ of unimodular $\boldsymbol{G}$-unitary matrices is pressed into service to represent the action of Lorentzian boosts, and inquire now into the question of whether or not that subgroup is in any way "distinguished." For an arbitrary 2×2 matrix $\boldsymbol{M}$ one has $\det(\boldsymbol{M}-\lambda\boldsymbol{I})=\lambda^2-\lambda\cdot\mathrm{tr}\boldsymbol{M}+\det\boldsymbol{M}$, so if $\boldsymbol{M}$ is traceless then the Cayley-Hamilton theorem supplies $\boldsymbol{M}^2+(\det\boldsymbol{M})\boldsymbol{I}=\boldsymbol{0}$, whence

$$e^{\boldsymbol{M}}=\cos\theta\cdot\boldsymbol{I}+\sin\theta\cdot\boldsymbol{M}/\theta \quad\text{with}\quad \theta\equiv\sqrt{\det\boldsymbol{M}}$$

In application of these general remarks we have

$$\boldsymbol{S}=e^{i\boldsymbol{H}}=\cos\theta\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sin\theta\cdot\frac{1}{\theta}\begin{pmatrix}s&iq\\ip&-s\end{pmatrix} \quad\text{with}\quad \theta\equiv\sqrt{pq-s^2}$$

which reproduces the design of (23.4). Boosts have been found to be associated with the case $p=q=0$, where we have

$$\begin{aligned}&=\cos is\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sin is\cdot\frac{1}{is}\begin{pmatrix}s&0\\0&-s\end{pmatrix}\\ &=\cosh s\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sinh s\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{aligned}$$

We recover (25) when, as stipulated at (21), we set $s = \frac{1}{2}\omega$. Let us now, at (24.6), set $p = u + v$ and $q = u - v$ so as to obtain

$$\boldsymbol{H}_{\text{traceless}} = \begin{pmatrix} -is & u-v \\ u+v & +is \end{pmatrix} \equiv s\boldsymbol{\tau}_0 + u\boldsymbol{\tau}_1 + v\boldsymbol{\tau}_2$$

Then

$$\det(s\boldsymbol{\tau}_0 + u\boldsymbol{\tau}_1 + v\boldsymbol{\tau}_2) = s^2 - u^2 + v^2$$

Each $\boldsymbol{\tau}$-matrix is traceless; their determinants are given by

$$\det\boldsymbol{\tau}_0 = +1; \quad \det\boldsymbol{\tau}_1 = -1; \quad \det\boldsymbol{\tau}_3 = +1$$

and their multiplicative properties present a kind of twisted version of the Pauli algebra:

$$\boldsymbol{\tau}_0^2 = -1; \quad \boldsymbol{\tau}_1^2 = +1; \quad \boldsymbol{\tau}_2^2 = -1$$

$$\begin{aligned} \boldsymbol{\tau}_0\boldsymbol{\tau}_1 &= -\boldsymbol{\tau}_1\boldsymbol{\tau}_0 = +i\boldsymbol{\tau}_2 \\ \boldsymbol{\tau}_1\boldsymbol{\tau}_2 &= -\boldsymbol{\tau}_2\boldsymbol{\tau}_1 = +i\boldsymbol{\tau}_0 \\ \boldsymbol{\tau}_2\boldsymbol{\tau}_0 &= -\boldsymbol{\tau}_0\boldsymbol{\tau}_2 = -i\boldsymbol{\tau}_1 \end{aligned}$$

If there is a "distinguished" element it would appear to be not $\boldsymbol{\tau}_0$ but $\boldsymbol{\tau}_1$. I had hoped to be able to assign meaning to the "spin" of the ψ-field, even though the toy theory does not support a concept of (orbital) angular momentum...but appear to be simply going in circles, so abandon this aspect of my topic. In the absence of a theory of spin it appears to be impossible to use Belinfante's trick (CFT, Chapter 2, p. 43) to achieve symmetrization of the stress-energy tensor.

Clifford algebras—especially the algebras of order 2. It appears to have been William Clifford who first undertook to take the "square root of a quadratic form," writing

$$p_1^2 + p_2^2 + \cdots + p_n^2 = (\boldsymbol{\epsilon}^1 p_1 + \boldsymbol{\epsilon}^2 p_2 + \cdots + \boldsymbol{\epsilon}^n p_n)^2 \tag{28}$$

and imposing upon the "hypernumbers" $\boldsymbol{\epsilon}^i$ the requirements that

$$\left.\begin{aligned} (\boldsymbol{\epsilon}^i)^2 &= 1 \quad : \quad i = 1, 2, \ldots, n \\ \boldsymbol{\epsilon}^i\boldsymbol{\epsilon}^j &= -\boldsymbol{\epsilon}^j\boldsymbol{\epsilon}^i \quad : \quad i \neq j \end{aligned}\right\} \tag{29}$$

though precisely why he did so—and where/when—remains unclear: persual of his *Collected Papers*[6] reveals an abiding interest in the geometric algebras of Hamilton and Grassmann, but it appears that what we now call "Clifford algebra" can be detected in Clifford's own work only as fragmentary hints (see the editor's remark at p. *lxvii* in the volume just cited), written near the end of his brief life (1845–1879). Pertti Lounestoe informs us[7] that Clifford's idea was

[6] The edition of 1882 was reprinted by Chelsea in 1968.
[7] *Clifford Algebras and Spinors* (1997), p. 9.

first mentioned in a talk (1876), the text of which was published posthumously as "XLIII. On the classification of geometrical algebras" in *Collected Papers*. Clifford's inspired murmuring attracted very little attention, however, until—fifty years later, and stripped of its geometrical overlay—the algebraic essence of the idea occurred independently to Dirac, and was put by him to stunningly productive work. The literature, some of it of intimidating complexity, has by become vast, and the theory of Clifford algebras (like quaternion algebra before it) has displayed a curious power to make fanatics of otherwise staid and sensible applied mathematicians and physicists. We will be concerned only with the simplest elements of the subject.[8]

Higher powers of the expression which appears squared on the right side of (28) are typified by the following:

$$\begin{aligned}&(\boldsymbol{\epsilon}^1 p_1 + \boldsymbol{\epsilon}^2 p_2 + \cdots + \boldsymbol{\epsilon}^n p_n)^7\\ &\qquad = \text{linear combination of terms of the type } \boldsymbol{\epsilon}^{i_1}\boldsymbol{\epsilon}^{i_2}\boldsymbol{\epsilon}^{i_3}\boldsymbol{\epsilon}^{i_4}\boldsymbol{\epsilon}^{i_5}\boldsymbol{\epsilon}^{i_6}\boldsymbol{\epsilon}^{i_7}\end{aligned}$$

Look to a typical such term: drawing upon (28) we have (on the presumption that $n \geq 9$)

$$\begin{aligned}\boldsymbol{\epsilon}^5\boldsymbol{\epsilon}^2\boldsymbol{\epsilon}^1\boldsymbol{\epsilon}^1\boldsymbol{\epsilon}^7\boldsymbol{\epsilon}^2\boldsymbol{\epsilon}^1 &= \underset{\uparrow\text{odd number of permutations required to achieve ascending order}}{-}\boldsymbol{\epsilon}^1\boldsymbol{\epsilon}^1\boldsymbol{\epsilon}^1\boldsymbol{\epsilon}^2\boldsymbol{\epsilon}^2\boldsymbol{\epsilon}^5\boldsymbol{\epsilon}^7\\ &= -\boldsymbol{\epsilon}^1\boldsymbol{\epsilon}^5\boldsymbol{\epsilon}^7 \quad \text{because} \begin{cases} (\boldsymbol{\epsilon}^i)^{\text{even}} = \mathbf{I} \\ (\boldsymbol{\epsilon}^i)^{\text{odd}} = \boldsymbol{\epsilon}^i \end{cases}\end{aligned}$$

Evidently every $\boldsymbol{\epsilon}$-product encountered in such an expression can, by (28), be brought to the form of one or another of the products which appear in the following list:

$$(\mathbf{I} \text{ else } \boldsymbol{\epsilon}^1)\cdot(\mathbf{I} \text{ else } \boldsymbol{\epsilon}^2)\cdots(\mathbf{I} \text{ else } \boldsymbol{\epsilon}^n)$$

The list asks us to make n binary choices, and therefore presents a total of 2^n distinct "canonical products." We are led thus to contemplate expressions of the design

$$\begin{aligned}\boldsymbol{A} = a\mathbf{I} + \sum_i a_i\boldsymbol{\epsilon}^i + \sum_{i<j} a_{ij}\boldsymbol{\epsilon}^i\boldsymbol{\epsilon}^j + \cdots + \sum_{i_1<i_2<\cdots<i_p} a_{i_1 i_2\cdots i_p}\boldsymbol{\epsilon}^{i_1}\boldsymbol{\epsilon}^{i_2}\cdots\boldsymbol{\epsilon}^{i_p}\\ + a_{12\cdots n}\boldsymbol{\epsilon}^1\boldsymbol{\epsilon}^2\cdots\boldsymbol{\epsilon}^n\end{aligned} \tag{30}$$

where the coefficients are taken to be (let us say) real numbers, and where there are evidently $\binom{n}{p}$ terms of order p. The set of all such "Clifford numbers" $\boldsymbol{A}$ is closed under both addition and multiplication, and is called the *Clifford algebra* $\mathcal{C}\ell_n$, of which $\{\boldsymbol{\epsilon}^1, \boldsymbol{\epsilon}^2, \ldots, \boldsymbol{\epsilon}^n\}$ are the "generators."

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[8] For good accounts see (for example) §§3–5 in P. K. Raševskiĭ, "The theory of spinors," American Mathematical Society Translations, Series 2, Volume 6 (1957) or the recent text by P. Lounestoe (cited above).

To place $\mathcal{C}\ell_n$ in its larger context: an *associative linear algebra* $\mathfrak{A}$ is a vector space—let the elements be notated

$$\boldsymbol{A} = a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + \cdots + a^m\boldsymbol{e}_m \equiv a^i\,\boldsymbol{e}_i$$

—on which a law of multiplication is defined

$$\begin{aligned}\boldsymbol{AB} &= a^i b^j\,\boldsymbol{e}_i\,\boldsymbol{e}_j\\ \boldsymbol{e}_i\,\boldsymbol{e}_j &\equiv \textstyle\sum_p c_i{}^p{}_j\,\boldsymbol{e}_p\end{aligned} \tag{31.1}$$

and is required, moreover, to be associative:

$$\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C} \quad\iff\quad c_i{}^q{}_p\,c_j{}^p{}_k = c_i{}^p{}_j\,c_p{}^q{}_k$$

Commutivity $(c_i{}^p{}_j = c_j{}^p{}_i)$ is, however, typically *not* required. If we define $\mathbb{E}_i \equiv \|c_i{}^q{}_p\|$ then the associativity condition can be expressed

$$\mathbb{E}_i\,\mathbb{E}_j = \textstyle\sum_p c_i{}^p{}_j\,\mathbb{E}_p \tag{31.2}$$

which shows that every associative algebra of order m admits of $m\times m$ matrix representation (this is the non-trivial converse of the trivial statement that every matrix algebra is associative), but leaves open the question of whether $\mathfrak{A}$ admits of lower-dimensional matrix representation. Notice that (31.1) is—insofar as it involves weighted summation—more general than would be permitted by stipulation that the $\boldsymbol{e}_i$ are elements of a finite group, though every such group provides a specialized instance of (31.1).

It is by now apparent that $\mathcal{C}\ell_n$ is a specialized associative linear algebra of order $m = 2^n$, and admits of $2^n\times 2^n$ matrix representation. It is, however, not presently clear how to construct the least-dimensional representation of $\mathcal{C}\ell_n$. Or how to construct $\boldsymbol{A}^{-1}$, or even how to decide (directly, without recourse to matrix representation theory) whether $\boldsymbol{A}^{-1}$ exists. These are typical of issues taken up in the literature.

Relativity—and a host of other pure/applied topics as well—inspire interest also in *indefinite* quadratic forms

$$p_1^2 + p_2^2 + \cdots + p_n^2 - q_1^2 - q_2^2 - \cdots - q_m^2$$

The associated Clifford algebras—which arise from writing

$$(\boldsymbol{\epsilon}^1 p_1 + \cdots + \boldsymbol{\epsilon}^n p_n + \boldsymbol{\epsilon}^{n+1}q_1 + \cdots + \boldsymbol{\epsilon}^{n+m}q_m)^2$$

and requiring that

$$\left.\begin{aligned}(\boldsymbol{\epsilon}^i)^2 &= \begin{cases}+\mathbf{I} & \text{if}\quad i = 1,2,\ldots,n\\ -\mathbf{I} & \text{if}\quad i = n+1, n+2,\ldots,m\end{cases}\\ \boldsymbol{\epsilon}^i\,\boldsymbol{\epsilon}^j &= -\boldsymbol{\epsilon}^j\,\boldsymbol{\epsilon}^i \quad \text{if}\quad i\neq j\end{aligned}\right\} \tag{32}$$

—are denoted $\mathcal{C}\ell_{n,m}$. The algebra previously designated $\mathcal{C}\ell_n$ would in this refined notation be designated $\mathcal{C}\ell_{n,0}$.

Evidently $\mathcal{C}\ell_{0,1}$ is just $\mathbb{C}$, the algebra of complex numbers, which we are in position now to observe admits of real 2×2 matrix representation. Look to the details: (32) reduces to the statement

$$\boldsymbol{\epsilon}^2 = -\mathsf{I}$$

Write $\mathsf{I} \to \boldsymbol{e}_1$ and $\boldsymbol{\epsilon} \to \boldsymbol{e}_2$ to establish contact with the generic language of (31.1). Then $\mathsf{I}{\cdot}\mathsf{I} = \mathsf{I}, \mathsf{I}{\cdot}\boldsymbol{\epsilon} = \boldsymbol{\epsilon}, \boldsymbol{\epsilon}{\cdot}\mathsf{I} = \boldsymbol{\epsilon}, \boldsymbol{\epsilon}{\cdot}\boldsymbol{\epsilon} = -\mathsf{I}$ read

$$\begin{aligned}
\boldsymbol{e}_1\boldsymbol{e}_1 &= {c_1}^1{}_1\boldsymbol{e}_1 + {c_1}^2{}_1\boldsymbol{e}_2 = 1\boldsymbol{e}_1 + 0\boldsymbol{e}_2\\
\boldsymbol{e}_1\boldsymbol{e}_2 &= {c_1}^1{}_2\boldsymbol{e}_1 + {c_1}^2{}_2\boldsymbol{e}_2 = 0\boldsymbol{e}_1 + 1\boldsymbol{e}_2\\
\boldsymbol{e}_2\boldsymbol{e}_1 &= {c_2}^1{}_1\boldsymbol{e}_1 + {c_2}^2{}_1\boldsymbol{e}_2 = 0\boldsymbol{e}_1 + 1\boldsymbol{e}_2\\
\boldsymbol{e}_2\boldsymbol{e}_2 &= {c_2}^1{}_2\boldsymbol{e}_1 + {c_2}^2{}_2\boldsymbol{e}_2 = -1\boldsymbol{e}_1 + 0\boldsymbol{e}_2
\end{aligned}$$

giving

$$\mathbb{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \mathbb{I} \qquad \text{and} \qquad \mathbb{E}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to \mathbb{J}$$

The elements $\boldsymbol{z} = x\mathsf{I} + y\boldsymbol{\epsilon}$ of $\mathcal{C}\ell_{0,1}$ acquire therefore the matrix representations

$$\mathbb{Z} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Evidently $\det\mathbb{Z} = x^2 + y^2$ and $\mathbb{Z}^{-1} \to (x^2+y^2)^{-1}(x\mathsf{I} - y\boldsymbol{\epsilon}) = (\boldsymbol{z}^*\boldsymbol{z})^{-1}\boldsymbol{z}^*$ with $\boldsymbol{z}^* = x\mathsf{I} - y\boldsymbol{\epsilon}$. I will, for methodological reasons, conclude with indication of how $\boldsymbol{z}^{-1}$ might have been obtained "directly, without recourse to matrix representation theory." Assume without significant loss of generality that $x = 1$ and write

$$\begin{aligned}
(\mathsf{I} + y\boldsymbol{\epsilon})^{-1} &= \mathsf{I} - y\boldsymbol{\epsilon} + (y\boldsymbol{\epsilon})^2 - (y\boldsymbol{\epsilon})^3 + (y\boldsymbol{\epsilon})^4 - (y\boldsymbol{\epsilon})^5 + \cdots\\
&= (1 - y^2 + y^4 - \cdots)\mathsf{I} - (y - y^3 + y^5 - \cdots)\boldsymbol{\epsilon}\\
&= (1 + y^2)^{-1}(\mathsf{I} - y\boldsymbol{\epsilon})
\end{aligned}$$

The argument fails only if $x = 0$, but in that case one has

$$(y\boldsymbol{\epsilon})^{-1} = (y^2)^{-1}(-y\boldsymbol{\epsilon})$$

by inspection.

The algebra $\mathcal{C}\ell_{0,2}$ is familiar as the algebra $\mathbb{Q}$ of real quaternions, as I now demonstrate. The general element can be written

$$\begin{aligned}
\boldsymbol{q} &= w\mathsf{I} + x\boldsymbol{\epsilon}_1 + y\boldsymbol{\epsilon}_2 + z\boldsymbol{\epsilon}_1\boldsymbol{\epsilon}_2\\
&= w\mathsf{I} + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \quad \text{in Hamilton's notation}
\end{aligned}$$

and the conditions (32) become

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -\mathsf{I} \tag{33.1}$$

$$\begin{aligned} \boldsymbol{ij} &= \boldsymbol{k} = -\boldsymbol{ji} \\ \boldsymbol{jk} &= \boldsymbol{i} = -\boldsymbol{kj} \\ \boldsymbol{ki} &= \boldsymbol{j} = -\boldsymbol{ik} \end{aligned} \tag{33.2}$$

of which

$$\begin{aligned} (a^1\boldsymbol{i} + a^2\boldsymbol{j} + a^3\boldsymbol{k})(b^1\boldsymbol{i} + b^2\boldsymbol{j} + b^3\boldsymbol{k}) = -(a^1b^1 + a^2b^2 + a^3b^3)\,&\mathsf{I} \\ + (a^2b^3 - a^3b^2)\,&\boldsymbol{i} \\ + (a^3b^1 - a^1b^3)\,&\boldsymbol{j} \\ + (a^1b^2 - a^2b^1)\,&\boldsymbol{k} \end{aligned}$$

is a celebrated particular consequence. We are assured that $\mathcal{C}\ell_{0,2}$ admits of real 4×4 matrix representation, but I skip the detailed demonstration in order to confront the force of the question: How, in the absence of an explicit matrix representation, does one construct $\boldsymbol{q}^{-1}$? Look to the series expansion of

$$[\mathsf{I} + (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})]^{-1}$$

and, using $(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})^2 = -(x^2 + y^2 + z^2)\,\mathsf{I}$ (established just above), obtain

$$[\mathsf{I} + (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})]^{-1} = (1 + x^2 + y^2 + z^2)^{-1}[\mathsf{I} - (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})]$$

Quick tinkering (send $x \to x/w, y \to y/w, z \to z/w$ and simplify) leads from here to the conclusion (valid even when $w = 0$) that $\boldsymbol{q}^{-1}$ exists if and only if $\boldsymbol{q}^*\boldsymbol{q} = w^2 + x^2 + y^2 + z^2 \neq 0$ (i.e., for all $\boldsymbol{q} \neq \boldsymbol{0}$) and is given then by

$$\boldsymbol{q}^{-1} = (\boldsymbol{q}^*\boldsymbol{q})^{-1}\boldsymbol{q}^* \quad \text{with} \quad \boldsymbol{q}^* \equiv w\,\mathsf{I} - x\boldsymbol{i} - y\boldsymbol{j} - z\boldsymbol{k}$$

A moment's further tinkering leads—not at all to our surprise—to this complex 2×2 representation of the quaterion algebra:

$$\boldsymbol{q} \to \mathbb{Q} = \begin{pmatrix} w + z & -x - iy \\ x - iy & w - z \end{pmatrix} \equiv w\,\mathbb{U} + x\,\mathbb{I} + y\,\mathbb{J} + z\,\mathbb{K}$$

We have

$$\det \mathbb{Q} = w^2 + x^2 + y^2 + z^2$$

and observe that the matrices $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ are traceless antihermitian. We recover the previously described representation of $\mathcal{C}\ell_{0,1}$ when $y = z = 0$.

$\mathcal{C}\ell_{1,3}$ is just the Dirac algebra—an algebra of order 16, generated by Dirac's celebrated $\boldsymbol{\gamma}^\mu$-matrices. It possesses a real 16×16 representation, but is well known to admit also of complex 4×4 representation.

Which brings us to $\mathcal{C}\ell_{1,1}$ —the "toy Dirac algebra" implicit in (1), of which we have been acquainted with the generators since (6) but have not previously had reason to regard as a closed algebra, with general element given by

$$\begin{aligned} \boldsymbol{d} = s\,\mathbf{I} + v_1\boldsymbol{\gamma}^1 + v_2\boldsymbol{\gamma}^2 + p\,\boldsymbol{\Gamma}& \\ \boldsymbol{\Gamma} \equiv \boldsymbol{\gamma}^1\boldsymbol{\gamma}^2 & \end{aligned} \tag{34}$$

$\mathcal{C}\ell_{1,1}$ is evidently a variant of the quaternion algebra; in place of (33) we have

$$(\boldsymbol{\gamma}^1)^2 = +\mathbf{I}\,, \quad (\boldsymbol{\gamma}^2)^2 = -\mathbf{I}\,, \quad (\boldsymbol{\Gamma})^2 = +\mathbf{I} \tag{35.1}$$

$$\begin{aligned} \boldsymbol{\gamma}^1\boldsymbol{\gamma}^2 &= \boldsymbol{\Gamma} \\ \boldsymbol{\gamma}^1\boldsymbol{\Gamma} &= \boldsymbol{\gamma}^2 \\ \boldsymbol{\gamma}^2\boldsymbol{\Gamma} &= \boldsymbol{\gamma}^1 \end{aligned} \tag{35.2}$$

A real 4×4 representation is assured, but a real (!) 2×2 representation is implicit in (6)

$$\boldsymbol{d} \to \mathbb{D} = \begin{pmatrix} s+p & v_1 - v_2 \\ v_1 + v_2 & s - p \end{pmatrix} \equiv s\,\mathbb{I} + v_1\,\mathit{\Gamma}^1 + v_2\,\mathit{\Gamma}^2 + p\,\mathit{\Gamma} \tag{36}$$

and is much easier to work with. Define

$$\mathbb{D}^* = s\,\mathbb{I} - v_1\,\mathit{\Gamma}^1 - v_2\,\mathit{\Gamma}^2 - p\,\mathit{\Gamma}$$

and obtain

$$\mathbb{D}^*\mathbb{D} = \underbrace{[s^2 - (v_1^2 - v_2^2) - p^2\,]}_{\det\mathbb{D}}\cdot\mathbb{I}$$

giving $\boldsymbol{d}^{-1} = (\boldsymbol{d}^*\boldsymbol{d})^{-1}\boldsymbol{d}^*$; this result is structurally quite familiar, but novel in respect to its detailed meaning.

The notations $s/v/p$ are intended to suggest *scalar/vector/pseudoscalar*, for reasons which I now discuss. Let

$$\boldsymbol{w} = a_1\boldsymbol{\gamma}^1 + a_2\boldsymbol{\gamma}^2 + b\,\boldsymbol{\Gamma} \quad \text{with} \quad a_1^2 - a_2^2 + b^2 = 1$$

Then $\boldsymbol{w}^2 = \mathbf{I}$ and

$$\boldsymbol{u} \equiv e^{\phi\boldsymbol{w}} = \cosh\phi\cdot\mathbf{I} + \sinh\phi\cdot\boldsymbol{w}$$

We look to the similarity transformation $\boldsymbol{d} \to \boldsymbol{D} = \boldsymbol{u}^{-1}\boldsymbol{d}\,\boldsymbol{u}$; i.e., to

$$\boldsymbol{D} = (\cosh\phi\cdot\mathbf{I} - \sinh\phi\cdot\boldsymbol{w})(s\,\mathbf{I} + v_1\boldsymbol{\gamma}^1 + v_2\boldsymbol{\gamma}^2 + p\,\boldsymbol{\Gamma})(\cosh\phi\cdot\mathbf{I} + \sinh\phi\cdot\boldsymbol{w})$$

which for infinitesimal ϕ becomes

$$\boldsymbol{d} \to \boldsymbol{D} = \boldsymbol{d} + \delta\phi\cdot[\,\boldsymbol{d}\,,\boldsymbol{w}\,] + \cdots$$

with

$$\begin{aligned}\delta\phi\cdot[\boldsymbol{d},\boldsymbol{w}] &= 2\delta\phi(bv_2-a_2p)\,\boldsymbol{\gamma}^1+2\delta\phi(bv_1-a_1p)\,\boldsymbol{\gamma}^2+2\delta\phi(a_2v_1-a_1v_2)\,\boldsymbol{\Gamma}\\ &= \delta v_1\boldsymbol{\gamma}^1+\delta v_2\boldsymbol{\gamma}^2+\delta p\,\boldsymbol{\Gamma}\end{aligned}$$

To exclude p-dependence from δv we find ourselves obliged to set $a_1=a_2=0$. Then $b=\pm1$; without loss of generality we set $b=1$, and obtain

$$\begin{aligned}\boldsymbol{D} &= e^{-\phi\boldsymbol{\Gamma}}\boldsymbol{d}\,e^{\phi\boldsymbol{\Gamma}}\\ &\qquad e^{\phi\boldsymbol{\Gamma}}=\cosh\phi\cdot\mathbf{I}+\sinh\phi\cdot\boldsymbol{\Gamma}\\ &= s\,\mathbf{I}+(v_1\cosh2\phi+v_2\sinh\phi)\,\boldsymbol{\gamma}^1+(v_1\sinh2\phi+v_2\cosh\phi)\,\boldsymbol{\gamma}^2+p\,\boldsymbol{\Gamma}\\ &= S\,\mathbf{I}+V_1\boldsymbol{\gamma}^1+V_2\boldsymbol{\gamma}^2+P\,\boldsymbol{\Gamma}\end{aligned}$$

We have here recovered the 2-dimensional Lorentz transformations as natural objects (and have at the same time indicated why it makes sense to use the term "vector" in reference to the v terms), but have left unexplained why p has been called a "pseudoscalar." Suppose, however, we were to set

$$\boldsymbol{u}=\boldsymbol{\gamma}^1\to\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad:\quad\text{in this case }\boldsymbol{u}^{-1}=\boldsymbol{u}$$

The unimodular (or $\boldsymbol{d}^*\boldsymbol{d}$-preserving) transformation $\boldsymbol{d}\to\boldsymbol{D}=\boldsymbol{u}^{-1}\boldsymbol{d}\,\boldsymbol{u}$ is then improper (disjoint from the identity), and we compute

$$\boldsymbol{D}=s\,\mathbf{I}+v_1\boldsymbol{\gamma}^1-v_2\boldsymbol{\gamma}^2-p\,\boldsymbol{\Gamma}$$

The Lorentz transformation $(v_1,v_2)\to(v_1,-v_2)$ is improper in the familiar sense, and has sent $p\to-p$.

Clifford algebra when the metric is non-diagonal. Familiar transformations serve to achieve

$$\begin{pmatrix}g_{11}&g_{12}&\cdots&g_{1N}\\g_{21}&g_{22}&\cdots&g_{2N}\\\vdots&\vdots&\ddots&\vdots\\g_{N1}&g_{N2}&\cdots&g_{NN}\end{pmatrix}\to\begin{pmatrix}g_1&&&\\&g_2&&\\&&\ddots&\\&&&g_N\end{pmatrix}\to\begin{pmatrix}\pm1&&&\\&\pm1&&\\&&\ddots&\\&&&\pm1\end{pmatrix}$$

where the final matrix has n plus signs on the diagonal, and $m=N-n$ minus signs. The theory of $\mathcal{C}\ell_{n,m}$ standardly assumes such preparations to have been carried out. I have (general relativistic) interest, however, in seeing how one might contrive to work with a non-specialized real symmetric metric. The algebra which arises from writing $g^{ij}p_ip_j=(\boldsymbol{\epsilon}^ip_i)^2$ or, as I find now more convenient,

$$g_{ij}x^ix^j=(x^i\boldsymbol{\epsilon}_i)^2 \tag{37}$$

will be designated $\mathcal{C}\ell_N(g)$, and springs from anticommutation relations

$$\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_j+\boldsymbol{\epsilon}_j\boldsymbol{\epsilon}_i=2g_{ij} \tag{38}$$

of which (29) no longer provide an accurate description. The only surviving allusion to the possibility that the metric might be "indefinite" will reside in the observation that

$$g \equiv \det \|g_{ij}\| = (-)^m |g| \tag{39}$$

The important role formerly played by the antisymmetry condition (29) will be taken over by the statement that, in consequence of (38),

$$\boldsymbol{\epsilon}_{ij} \equiv \tfrac{1}{2}(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_j - \boldsymbol{\epsilon}_j\boldsymbol{\epsilon}_i) = \boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_j - g_{ij}\,\mathbf{I} \quad \text{is antisymmetric} \tag{40}$$

In 1960 I had reason to work out the detailed theory of $\mathcal{C}\ell_4(g)$. Here, taking that work[9] as my model, I will look only to the simplest case: $\mathcal{C}\ell_2(g)$.

The general element of $\mathcal{C}\ell_2(g)$ can be written

$$\boldsymbol{G} = S\,\mathbf{I} + V^i\boldsymbol{\epsilon}_i + \tfrac{1}{2}T^{ij}\boldsymbol{\epsilon}_{ij} \tag{41}$$

where T^{ij} can without loss of generality be assumed to be antisymmetric; since 2-dimensional, it supplies only a single adjustable parameter, so the final term on the right could equally well be expressed $T^{12}\boldsymbol{\epsilon}_{12}$ or—more simply—$P\boldsymbol{\epsilon}$. So we write

$$\boldsymbol{G} = S\,\mathbf{I} + V^i\boldsymbol{\epsilon}_i + P\boldsymbol{\epsilon} \tag{42}$$

If

$$\boldsymbol{H} = s\,\mathbf{I} + v^i\,\boldsymbol{\epsilon}_i + p\,\boldsymbol{\epsilon}$$

describes an arbitrary second element in $\mathcal{C}\ell_2(g)$ then a little work supplies the product formula

$$\begin{aligned}\boldsymbol{GH} = (Ss + V^i v_i - gPp)\,\mathbf{I} &+ (Sv^j + sV^j - Pv_i\varepsilon^{ij} + pV_i\varepsilon^{ij})\,\boldsymbol{\epsilon}_j \\ &+ (Sp + V^1v^2 - V^2v^1 + Ps)\,\boldsymbol{\epsilon}\end{aligned} \tag{43}$$

where it is actually the expressions on the right sides of

$$\begin{aligned} V^i v_i &= V^1 g_{11} v^1 + V^1 g_{12} v^2 + V^2 g_{21} v^1 + V^2 g_{22} v^2 \\ g &= g_{11}g_{22} - g_{12}g_{21} \\ V_i &= g_{i1}V^1 + g_{i2}V^2 \\ v_i &= g_{i1}\,v^1 \ + g_{i2}\,v^2 \end{aligned}$$

which are presented to us in the course of the calculation, and where

$$\varepsilon^{ij} \equiv \text{sgn}\binom{i\,j}{1\,2}$$

I strongly urge my reader to *do* the calculation that gives (43); it takes only a few minutes of careful work on a large sheet of paper, and is highly instructive.

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[9] See "Aspects of Clifford algebras" (text of a seminar presented on 27 March 1967 to the Reed College Math Club—back in the days before there were such things Thursday Math Seminars) in COLLECTED SEMINARS (1963–1970).

If we define

$$\boldsymbol{G}^* = S\mathbf{I} - V^i\boldsymbol{\epsilon}_i - P\boldsymbol{\epsilon} \tag{44}$$

then it is an immediate implication of (43) that

$$\boldsymbol{G}^*\boldsymbol{G} = (S^2 - g_{ij}V^iV^j + gP^2)\mathbf{I}$$

Evidently $\boldsymbol{G}$ is invertible if and only if its "modulus"

$$|\boldsymbol{G}| \equiv S^2 - g_{ij}V^iV^j + gP^2 \neq 0 \tag{45}$$

In a matrix representation $\boldsymbol{G} \to \mathbb{G}$ the role of the modulus would be taken over by $\det\mathbb{G}$. We find it natural, therefore, to construct the

$$\text{“characteristic polynomial”}\;\; |\boldsymbol{G} - \lambda\mathbf{I}| = \lambda^2 - 2S\lambda + |\boldsymbol{G}|$$

and are not surprised to discover that $\boldsymbol{G}$ satisfies its own characteristic equation:

$$\boldsymbol{G}^2 - 2S\boldsymbol{G} + |\boldsymbol{G}| = \mathbf{0} \tag{46}$$

We are assured that $\mathcal{C}\ell_2(g)$—since an associative algebra of order 4—admits of real 4×4 representation, but the preceding results suggest strongly that it admits actually of 2×2 representation.

The Clifford algebras $\mathcal{C}\ell_{2,0}$, $\mathcal{C}\ell_{1,1}$ and $\mathcal{C}\ell_{0,2}$ can be recovered as special cases of $\mathcal{C}\ell_2(g)$. To see how this works we look to the latter; i.e., to the quaternionic case

$$\|g_{ij}\| = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

After notational adjustments

$$\begin{array}{lll} S \;\to a^0 & s \;\to b^0 & \\ V^1 \to a^1 & v^1 \to b^1 & \boldsymbol{\epsilon}_1 \to \boldsymbol{i} \\ V^2 \to a^2 & v^2 \to b^2 & \boldsymbol{\epsilon}_2 \to \boldsymbol{j} \\ & & \boldsymbol{\epsilon} \;\to \boldsymbol{k} \\ P \;\to a^3 & p \;\to b^3 & \end{array}$$

the general product formula (43) becomes

$$\begin{aligned}(a^0\mathbf{I} + a^1\boldsymbol{i} + a^2\boldsymbol{j} + a^3\boldsymbol{k})&(b^0\mathbf{I} + b^1\boldsymbol{i} + b^2\boldsymbol{j} + b^3\boldsymbol{k}) \\ &= (a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3)\mathbf{I} + (a^0b^1 + a^1b^0 + a^2b^3 - a^3b^2)\boldsymbol{i} \\ &\qquad + (a^0b^2 + a^2b^0 + a^3b^1 - a^1b^3)\boldsymbol{j} \\ &\qquad + (a^0b^3 + a^3b^0 + a^1b^2 - a^2b^1)\boldsymbol{k}\end{aligned}$$

which is familiar as the *quaternionic product formula*—historic birthplace of the "dot product," the "cross product" (set $a^0 = b^0 = 0$) and of non-commutative algebra generally.

A notational adjustment

$$\mathsf{I} \to \boldsymbol{e}_0\ ;\quad \boldsymbol{\epsilon}_1 \to \boldsymbol{e}_1\ ;\quad \boldsymbol{\epsilon}_2 \to \boldsymbol{e}_2\ ;\quad \boldsymbol{\epsilon} \to \boldsymbol{e}_3$$

places us in position to use (31) to construct a real 4×4 representation of $\mathcal{C}\ell_2(g)$. Working from (43) we have

$$\begin{aligned}
\boldsymbol{e}_0(s\boldsymbol{e}_0 + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + p\boldsymbol{e}_3) &= s\boldsymbol{e}_0 + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + p\boldsymbol{e}_3\\
\boldsymbol{e}_1(s\boldsymbol{e}_0 + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + p\boldsymbol{e}_3) &= g_{1k}v^k\boldsymbol{e}_0 + (s - pg_{12})\boldsymbol{e}_1 + pg_{11}\boldsymbol{e}_2 + v^2\boldsymbol{e}_3\\
\boldsymbol{e}_2(s\boldsymbol{e}_0 + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + p\boldsymbol{e}_3) &= g_{2k}v^k\boldsymbol{e}_0 - pg_{22}\boldsymbol{e}_1 + (s + pg_{21})\boldsymbol{e}_2 - v^1\boldsymbol{e}_3\\
\boldsymbol{e}_3(s\boldsymbol{e}_0 + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + p\boldsymbol{e}_3) &= -gp\boldsymbol{e}_0 + v^i g_{i2}\boldsymbol{e}_1 - v^i g_{i1}\boldsymbol{e}_2 + s\boldsymbol{e}_3
\end{aligned}$$

giving

$$\mathsf{I} = \boldsymbol{e}_0 \to \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{47.0}$$

$$\boldsymbol{\epsilon}_1 = \boldsymbol{e}_1 \to \mathbb{E}_1 = \begin{pmatrix} 0 & g_{11} & g_{12} & 0\\ 1 & 0 & 0 & -g_{12}\\ 0 & 0 & 0 & g_{11}\\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{47.1}$$

$$\boldsymbol{\epsilon}_2 = \boldsymbol{e}_2 \to \mathbb{E}_2 = \begin{pmatrix} 0 & g_{21} & g_{22} & 0\\ 0 & 0 & 0 & -g_{22}\\ 1 & 0 & 0 & g_{21}\\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{47.2}$$

$$\boldsymbol{\epsilon} = \boldsymbol{e}_3 \to \mathbb{E} = \begin{pmatrix} 0 & 0 & 0 & -g\\ 0 & g_{12} & g_{22} & 0\\ 0 & -g_{11} & -g_{21} & 0\\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{47.3}$$

Implausible though the design of these matrices may appear, *Mathematica* quickly confirms that

$$\mathbb{E}_i\mathbb{E}_j + \mathbb{E}_j\mathbb{E}_i = 2g_{ij}\mathbb{I} \quad\text{and}\quad \mathbb{E}_1\mathbb{E}_2 - g_{12}\mathbb{I} = \mathbb{E} \tag{48}$$

The representation of $\boldsymbol{G} = S\mathsf{I} + V^i\boldsymbol{\epsilon}_i + P\boldsymbol{\epsilon}$ becomes

$$\begin{aligned}
\mathbb{G} &= S\mathbb{I} + V^1\mathbb{E}_1 + V^2\mathbb{E}_2 + P\,\mathbb{E}\\
&= \begin{pmatrix} S & V_1 & V_2 & -Pg\\ V^1 & S{+}Pg_{12} & Pg_{22} & -V_2\\ V^2 & -Pg_{11} & S{-}Pg_{21} & V_1\\ P & -V^2 & V^1 & S \end{pmatrix}
\end{aligned} \tag{49}$$

and, appealing again to *Mathematica* for assistance, we find

$$\det\mathbb{G} = (S^2 - V^iV_i + gP^2)^2 = |\boldsymbol{G}|^2 \tag{50}$$

If, in (49), we set $S = P = 0$ we arrive back at our point of departure

$$\begin{aligned}(V^1\mathbb{E}_1 + V^2\mathbb{E}_2)^2 &= \begin{pmatrix} 0 & V_1 & V_2 & 0\\ V^1 & 0 & 0 & -V_2\\ V^2 & 0 & 0 & V_1\\ 0 & -V^2 & V^1 & 0\end{pmatrix}^2\\ &= (V_1V^1 + V_2V^2)\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\end{aligned}$$

. . . which is comforting, if not surprising.

It should be borne in mind that the representation (47/49) is not unique: similarity transformations

$$\mathbb{E}_\mu \to \mathbb{E}_\mu \equiv \mathbb{T}^{-1}\mathbb{E}_\mu\mathbb{T} \tag{51}$$

yield alternative/equivalent representations.

A "real 4-component formulation of a toy Dirac theory with arbitrary metric" is our present objective, but before we can make further progress we must gain a sharper sense of certain general features shared by all "real-field Dirac theories," and of the conditions under which such theories admit of Lagangian formulation. We must, in particular, discover what becomes of the i which enters so conspicuously into the Dirac equation, but can have no place in a real-field theory. To get a handle on the points at issue we look back again to (5); i.e., to our toy theory in the Lorentzian case. If we

$$\text{let } \psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix}\alpha^1 + i\beta^1\\ \alpha^2 + i\beta^2\end{pmatrix} \text{ be expanded } \begin{pmatrix}\alpha^1\\ \beta^1\\ \alpha^2\\ \beta^2\end{pmatrix}$$

then (5) becomes (note the absence of i-factors!)

$$\left[\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\partial_0 + \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\partial_1 + \varkappa\begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\right]\begin{pmatrix}\alpha^1\\ \beta^1\\ \alpha^2\\ \beta^2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

which will be abbreviated

$$\big[\boldsymbol{\Gamma}^0\partial_0 + \boldsymbol{\Gamma}^1\partial_1 + \varkappa\,\mathbb{J}\,\big]\Psi = 0 \tag{52.1}$$

We observe that

$$\boldsymbol{\Gamma}^\mu\boldsymbol{\Gamma}^\nu + \boldsymbol{\Gamma}^\nu\boldsymbol{\Gamma}^\mu = 2g^{\mu\nu}\,\mathbb{I} \quad\text{with}\quad \|g^{\mu\nu}\| = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{52.2}$$

and—which are more to the point—that

$$\mathbb{J} \text{ commutes with } \boldsymbol{\Gamma}^0 \text{ and } \boldsymbol{\Gamma}^1\text{; moreover } \mathbb{J}^2 = -\mathbb{I} \tag{52.3}$$

In the complex Dirac theory of the textbooks one achieves (52.3) by setting $\mathbb{J} = i\,\mathbb{I}$, but that option is not in all cases forced. . . and in real theory not possible; $\mathbb{J}$ is a "square root of minus $\mathbb{I}$" in an enlarged, matrix-theoretic sense.

If (52.1) is to admit of Lagrangian formulation then there must exist a real non-singular matrix $\mathbb{S}$ such that

$$\mathbb{S}\boldsymbol{\mathit{\Gamma}}^0 \text{ and } \mathbb{S}\boldsymbol{\mathit{\Gamma}}^1 \text{ are antisymmetric, and } \mathbb{S}\mathbb{J} \text{ symmetric} \tag{52.4}$$

in which case we have[10]

$$\mathcal{L} = -\tfrac{1}{2}\hbar c\big\{\Psi^{\mathsf{T}}\mathbb{S}\boldsymbol{\mathit{\Gamma}}^\mu\Psi_{,\mu} + \varkappa\Psi^{\mathsf{T}}\mathbb{S}\mathbb{J}\,\Psi\big\} \tag{52.5}$$

Exploratory tinkering (I know of no systematic method) shows that a $\mathbb{S}$ that does the job is

$$\mathbb{S} \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{52.6}$$

for

$$\mathbb{S}\boldsymbol{\mathit{\Gamma}}^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{S}\boldsymbol{\mathit{\Gamma}}^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

are antisymmetric, while

$$\mathbb{S}\mathbb{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is symmetric.

Look now to the generality of the points at issue. The "Dirac factorization problem"

$$(\boldsymbol{\gamma}^\mu\partial_\mu - \varkappa\,\mathbf{J})(\boldsymbol{\gamma}^\mu\partial_\mu + \varkappa\,\mathbf{J}) = (\square + \varkappa^2)\,\mathbf{I} \tag{53.1}$$

[10] Notice that the symmetric part of $\mathbb{S}\boldsymbol{\mathit{\Gamma}}^\mu$, if present, could be discarded as a gauge term. And that while in quantum theory with complex ψ it is permissible (and standard) to treat ψ and ψ^* as though they were formally independent, it would be senseless to assign "formal independence" to Ψ and Ψ^{T}. Only when these points are understood does it become permissible (if not very useful) to write

$$\tilde{\Psi} \equiv \Psi^{\mathsf{T}}\mathbb{S}$$

requires, in addition to the familiar conditions $\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu = 2g^{\mu\nu}\mathbf{I}$, that

$$\mathbf{J} \text{ commutes with all } \boldsymbol{\gamma}\text{-matrices; moreover } \mathbf{J}^2 = -\mathbf{I} \tag{53.2}$$

From $[\mathbf{J}, \boldsymbol{\gamma}^\mu] = \mathbf{0}$ it follows that $\mathbf{J}$ commutes with all elements of the Clifford algebra *generated* by the $\boldsymbol{\gamma}$-matrices; i.e., that

$$\mathbf{J} \in \text{ the ``center'' of the algebra} \tag{53.3}$$

In standard complex-field Dirac theory the center contains only multiples of $\mathbf{I}$, and the introduction of i-factors is forced. But in real-field Dirac theory the center may/must contain additional elements. The Dirac equation

$$(\boldsymbol{\gamma}^\mu\partial_\mu + \varkappa\mathbf{J})\Psi = 0 \tag{53.4}$$

admits of Lagrangian formulation if and only if there exists an invertible $\boldsymbol{S}$ such that

$$\boldsymbol{S}\boldsymbol{\gamma}^\mu \text{ is antisymmetric (all } \mu) \text{ and } \boldsymbol{S}\mathbf{J} \text{ is symmetric} \tag{53.5}$$

in which case one has

$$\mathcal{L} = -\tfrac{1}{2}\hbar c\big\{\Psi^\intercal\boldsymbol{S}\boldsymbol{\gamma}^\mu\Psi_{,\mu} + \varkappa\Psi^\intercal\boldsymbol{S}\mathbf{J}\Psi\big\} \tag{53.6}$$

But what we presently lack are constructive means to *exhibit* matrices $\mathbf{J}$ and $\boldsymbol{S}$ with the requisite properties, or even guarantee of their existence. That this is a major handicap will soon be evident:

In the Lorentzian case the regular representation formulae (47.1) and (47.2) supply[11]

$$\boldsymbol{\varGamma}^1 = g^{1k}\,\mathbb{E}_k = +\mathbb{E}_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

$$\boldsymbol{\varGamma}^2 = g^{2k}\,\mathbb{E}_k = -\mathbb{E}_2 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$$

which when combined with $\boldsymbol{\varGamma} \equiv \boldsymbol{\varGamma}^{12} \equiv \boldsymbol{\varGamma}^1\boldsymbol{\varGamma}^2 - g^{12}\,\mathbb{I} = \boldsymbol{\varGamma}^1\boldsymbol{\varGamma}^2$ entail

$$\begin{aligned}\mathbb{G} &= S\,\mathbb{I} + V_1\boldsymbol{\varGamma}^1 + V_2\boldsymbol{\varGamma}^2 + P\boldsymbol{\varGamma}\\ &= \begin{pmatrix} S & V_1 & V_2 & -P\\ V_1 & S & P & -V_2\\ -V_2 & P & S & V_1\\ -P & V_2 & V_1 & S\end{pmatrix}\end{aligned}$$

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[11] The following remarks are intended to be read on-screen, and will be rendered confusing by loss of the colored typography which I have used to avoid confusing hats, primes, etc.

Notice that $\boldsymbol{\Gamma}^0$ and $\boldsymbol{\Gamma}^1$ differ from the $\boldsymbol{\Gamma}^0$ and $\boldsymbol{\Gamma}^1$ introduced at (52.1). I have not been able to establish their equivalence in the sense (51), and suspect that they are *not* equivalent. Nor have I been able to discover either a $\mathbb{J}$ which commutes with $\boldsymbol{\Gamma}^0$ and $\boldsymbol{\Gamma}^1$ or a $\mathbb{S}$ which renders $\mathbb{S}\boldsymbol{\Gamma}^0$ and $\mathbb{S}\boldsymbol{\Gamma}^1$ antisymmetric (but neither have I been able to show that such things are impossible). Multiple failure in this relatively simple case leads me to think that it will be difficult/ impossible to erect a "Lagrangian formulation of real 4-component toy Dirac theory with general metric" on the platform provided by (49).

But why are we interested in real 4-component theory? Only because the regular representation of $\mathcal{C}\ell_2(g)$ has supplied real 4×4 matrices $\boldsymbol{\Gamma}^0$ and $\boldsymbol{\Gamma}^1$. In the thought that "general metric theory" should be constructed on some alternative pattern, we observe that if we introduce

$$\mathbb{H} \equiv \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad : \quad \text{hermitian} \tag{54.1}$$

then (49) supplies

$$\mathbb{H}\mathbb{G} = S\,\mathbb{H} + i\underbrace{\begin{pmatrix} P & -V^2 & +V^1 & 0 \\ -V^2 & Pg_{11} & Pg_{12} & -V_1 \\ +V^1 & Pg_{21} & Pg_{22} & -V_2 \\ 0 & -V_1 & -V_2 & Pg \end{pmatrix}}_{\text{antihermitian!}} \tag{54.2}$$

The pretty implication is that if we take (47.1) and (47.2) to be definitions of $\boldsymbol{\Gamma}_0 \equiv \mathbb{E}_0$ and $\boldsymbol{\Gamma}_1 \equiv \mathbb{E}_1$ then

$$(\boldsymbol{\Gamma}_\mu\partial^\mu - i\varkappa\,\mathbb{I})(\boldsymbol{\Gamma}_\nu\partial^\nu + i\varkappa\,\mathbb{I}) = (\Box + \varkappa^2)\,\mathbb{I}$$
$$\Box \equiv g_{\mu\nu}\partial^\mu\partial^\nu \text{ with general metric}$$

which motivates us to write

$$(\boldsymbol{\Gamma}_\mu\partial^\mu + i\varkappa\,\mathbb{I})\psi = 0 \quad \text{with} \quad \psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} \tag{55}$$

and to observe that this "toy Dirac equation with general metric" can be obtained from a Lagrangian of (compare (9.0)) this classic design:

$$\mathcal{L} = -\hbar c\Big[i\tfrac{1}{2}\big\{\psi^\dagger_{,\mu}\mathbb{H}\boldsymbol{\Gamma}^\mu\psi - \psi^\dagger\mathbb{H}\boldsymbol{\Gamma}^\mu\psi_{,\mu}\big\} + \varkappa\,\psi^\dagger\mathbb{H}\psi\Big] \tag{56}$$

Were one to separate the real from the imaginary components of ψ one would be led in the toy theory to an 8-component wave function Ψ (and in the physical case studied by Dirac to a 32-component wave function!).

Anticipated continuation. In the 4-component theory (55) the $\mathit{\Gamma}$-matrices are 4×4 and real. When I have an opportunity to resume this discussion I will attempt to construct a 2×2 complex realization of $\mathcal{C}\ell_2(g)$. While it was established by Dirac that $\mathcal{C}\ell_{1,3}$ admits of 4×4 complex realization, it is not clear that $\mathcal{C}\ell_4(g)$ does; if so, then one might possibly expect to have a 3×3 complex realization $\mathcal{C}\ell_3(g)$ of, even though the regular realization is 8×8. It is my experience, however, that—in this area especially—numerology is an unreliable guide.

I propose to explore also the following topics:

• Foldy-Wouthuysen representation in 2-dimensional theory (take maybe Schweber's §4f as my point of departure).

• Massless Dirac fields in 2-dimensional theory.

• Abelian/non-Abelian gauge field theories supported by the 2-dimensional Dirac theory.

• Dirac theory on curved 2-dimensional manifolds $g_{\mu\nu}(x)$. The "Vierbein formalism" becomes a "Zweibein formalism"? Use general covariance to achieve symmetrization of the stress energy tensor?

• At (54.2) we encounter (set $S = V^1 = V^2 = 0$ and $P = 1$) a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & g \end{pmatrix}$$

encountered also in exterior algebra (See ELECTRODYNAMICS (1972), p. 156). It would be instructive to develop the connection (which is well known to people like Lounesto).